Theoretical Investigation of the particle response to an acoustic field

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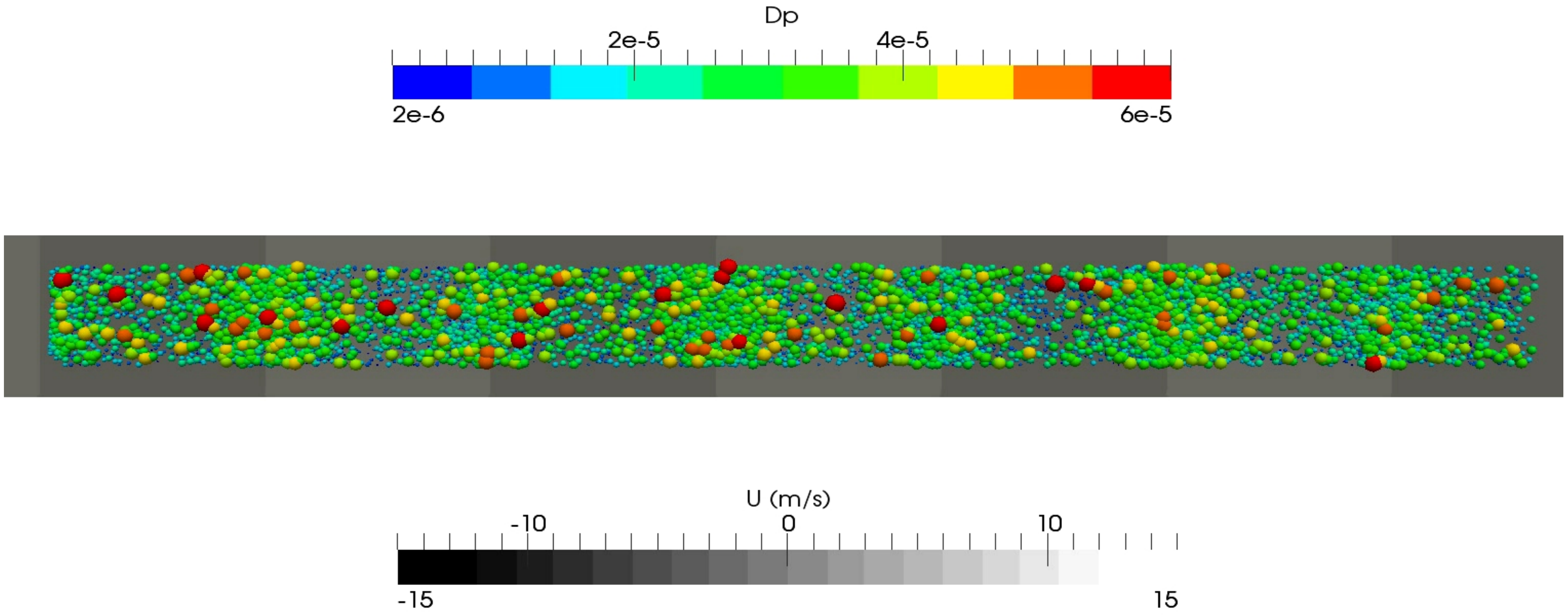
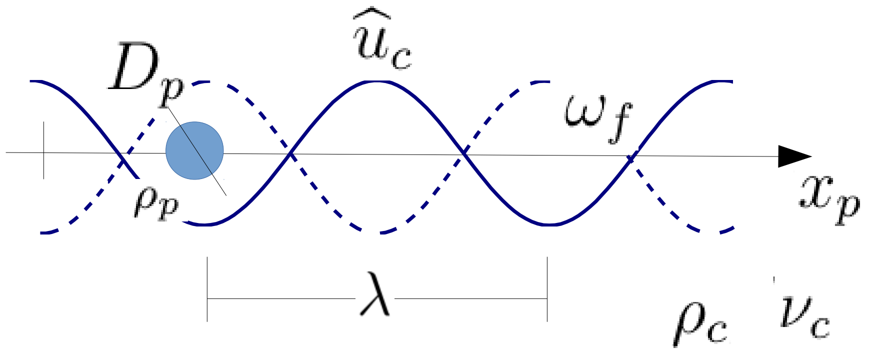
Motivation

Agglomeration induced by an acoustic standing wave.

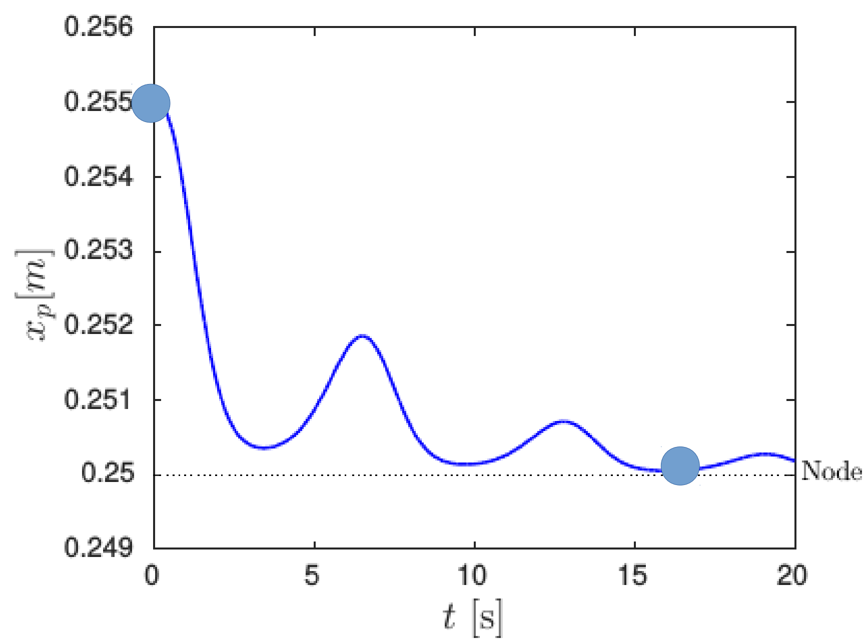


Objective

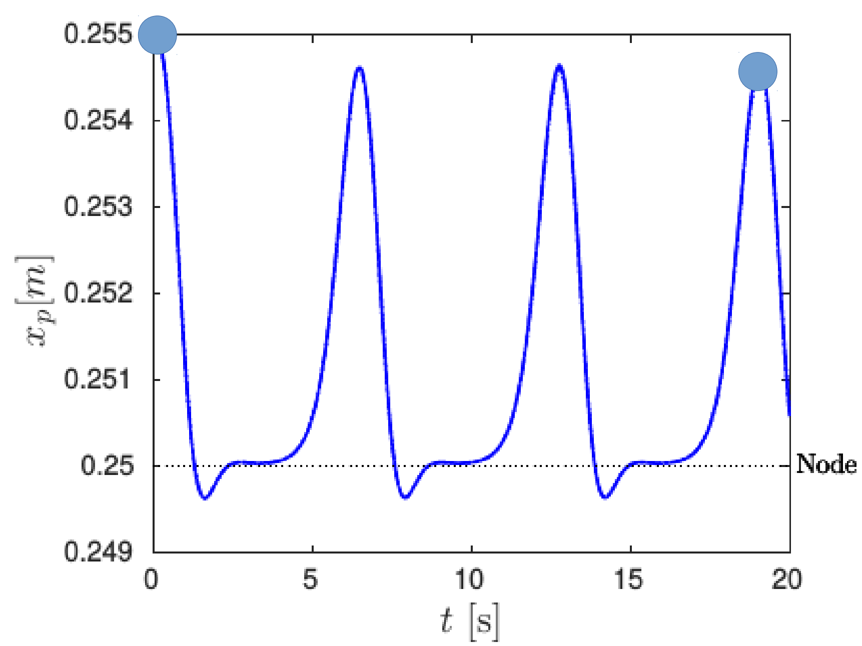
System parameters.



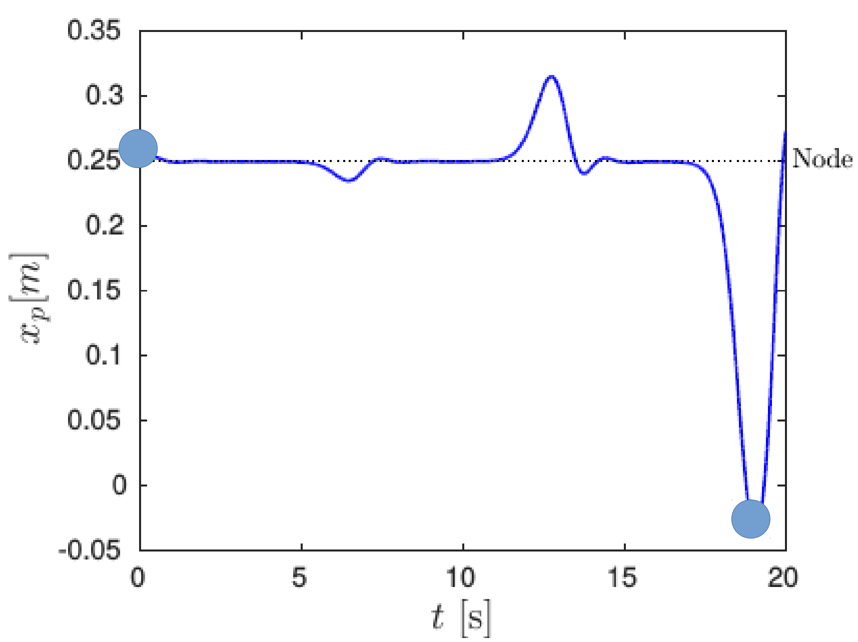
A different behavior can be found for different combination of parameters?



Stable



Periodic



Unstable (resonant)

Outline

- Governing equation.
- Linearization and theoretical approach for the solution by a complex Fourier series.
- Parametric stability of the solution and development of stability maps.
- Validation and conclusions.

Governing equation

Balance of forces

$$\ddot{x}_p + f_1 C \dot{x}_p = f_1 C u_c + \frac{1}{\gamma}\frac{Du_c}{Dt}$$

$$C = \frac{18\nu_c}{\gamma D_p^2}$$

$$\gamma = \rho_p/\rho_c$$

Non-linearities

(1) $$f_1 = \begin{cases} 1 & \rightarrow \text{Stokes flow. } Re_p < 1 \\ 1 + 0.15 Re_p^{0.687} & \rightarrow \text{Schiller and Naumman. } Re_p < 800 \end{cases}$$

$$Re_p = |u_c - \dot{x}_p| D_p/\nu_c$$

(2) $$u_c(x_p, t) = \widehat{u}_c \sin(\omega_f t)\cos(\tfrac{2\pi}{\lambda} x_p)$$

This non-linearity is the particularity of our work

Linearization

Non-linear system
(Stokes flow)

$$\ddot{x}_p + C\dot{x}_p = Cu_c + \frac{1}{\gamma}\frac{Du_c}{Dt}$$

Linearized around the
equilibrium point
(Standing wave node)

$$\ddot{x}_p + C\dot{x}_p + \underbrace{f(t)}_{\text{spring coeff.}} x_p = 0$$

$$f(t) = \frac{2\pi\widehat{u}_c C}{\lambda}\sin(\omega_f t) + \frac{1}{\gamma}\frac{2\pi\widehat{u}_c f}{\lambda}\cos(\omega_f t) - \frac{1}{\gamma}\left(\frac{2\pi\widehat{u}_c}{\lambda}\right)^2\sin^2(\omega_f t)$$

parameters reduction

$S_t = \omega_f / 2\pi C$ Stokes number

$\Delta = 2\pi\widehat{u}_c / \lambda\omega_f$ Acoustic displacement

$\gamma = \rho_p / \rho_c$ Ratio of densities

Hill's Equation

the variable transformation

$$q(t) = x_p(t)e^{\frac{C}{2}t}$$

$$\tau = \omega_f t$$

Non-dimensional time

produces a Hill's equation

$$\ddot{q}(\tau) + \left(- \left(\frac{1}{4\pi S_t} \right)^2 + \frac{\Delta}{2\pi S_t} \sin(\tau) + \frac{\Delta}{\gamma} \cos(\tau) + \frac{\Delta^2}{\gamma} \sin^2(\tau) \right) q(\tau) = 0$$

Parameter

Periodic function

Solution in form of a complex Fourier series

Critical growth rate

$$x_p(t) = q(t)\ e^{-\frac{C}{2}t}$$

a real growth/decay rate is added to this mode

the first sub-harmonic is dominant

$$q(\tau) = \sum_{n=-\infty}^{n=+\infty} c_n e^{\left(\frac{\omega_n}{\omega_f}+i\frac{n}{2}\right)\tau}$$

$$\omega_1 = C/2 \qquad\qquad \omega_n = 0 \text{ for } n \neq 1$$

Inserted in the differential equation

$$\sum_{n=-\infty}^{n=+\infty} c_n e^{\left(\frac{\omega_n}{\omega_f}+i\frac{n}{2}\right)\tau}\left(\left(\frac{\omega_n}{\omega_f}+i\frac{n}{2}\right)^2-\left(\frac{1}{4\pi S_t}\right)^2+\frac{\Delta}{4\pi S_t}\left(\frac{e^{i\tau}-e^{-i\tau}}{2i}\right)+\frac{\Delta}{\gamma}\left(\frac{e^{i\tau}-e^{-i\tau}}{2}\right)-\frac{\Delta^2}{4\gamma}\left(e^{2i\tau}-e^{-2i\tau}-2\right)\right)=0$$

recursive relation is found

$$\frac{\Delta^2}{4\gamma}c_{n-4}+\left(-\frac{i\Delta}{8\pi S_t}-\frac{\Delta}{2\gamma}\right)c_{n-2}+\left(\frac{-in}{4\pi S_t}-\frac{n^2}{4}-\frac{\Delta^2}{2\gamma}\right)c_n+\left(\frac{i\Delta}{8\pi S_t}-\frac{\Delta}{2\gamma}\right)c_{n+2}+\frac{\Delta^2}{4\gamma}c_{n+4}=0$$

Hill's determinant

$$H_N c_n = 0$$

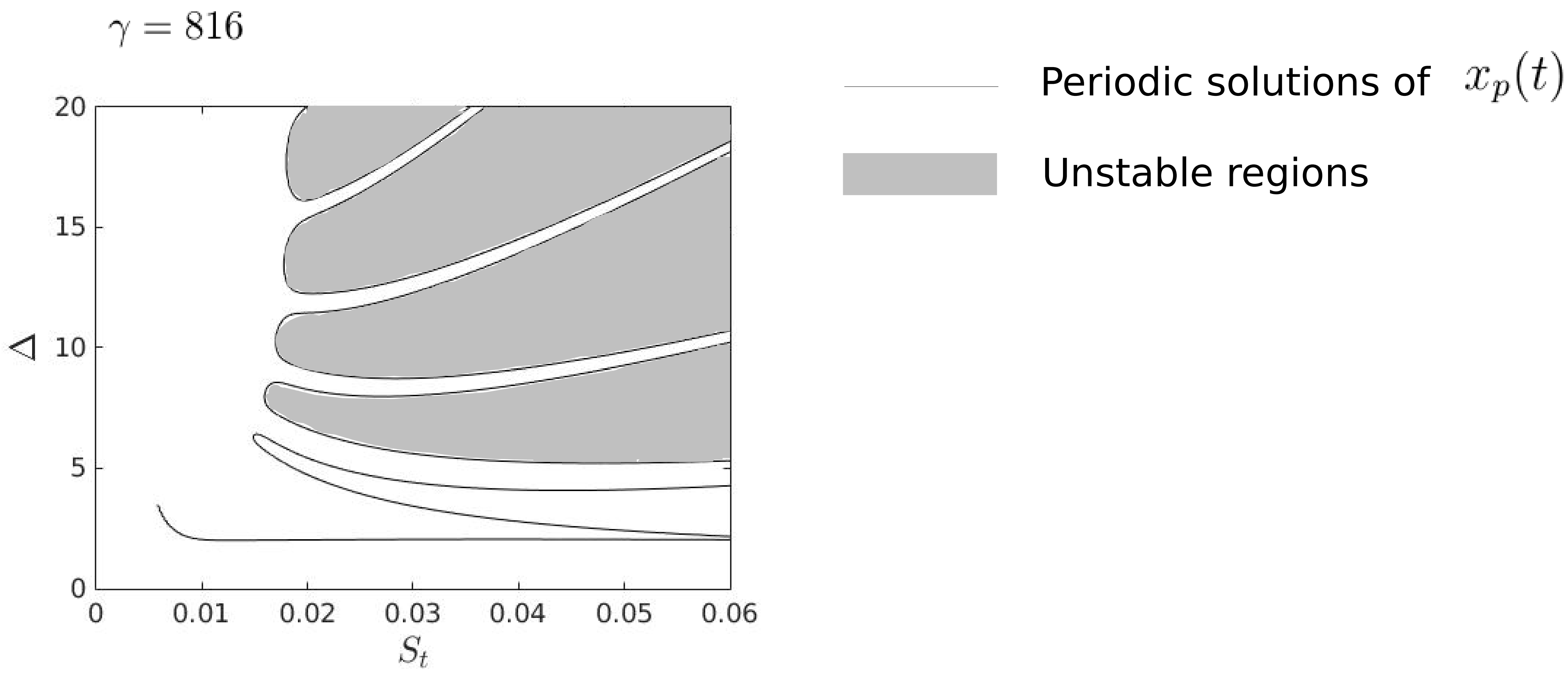
No trivial solution if $det(H_N) = 0$

which is an implicit polynomial

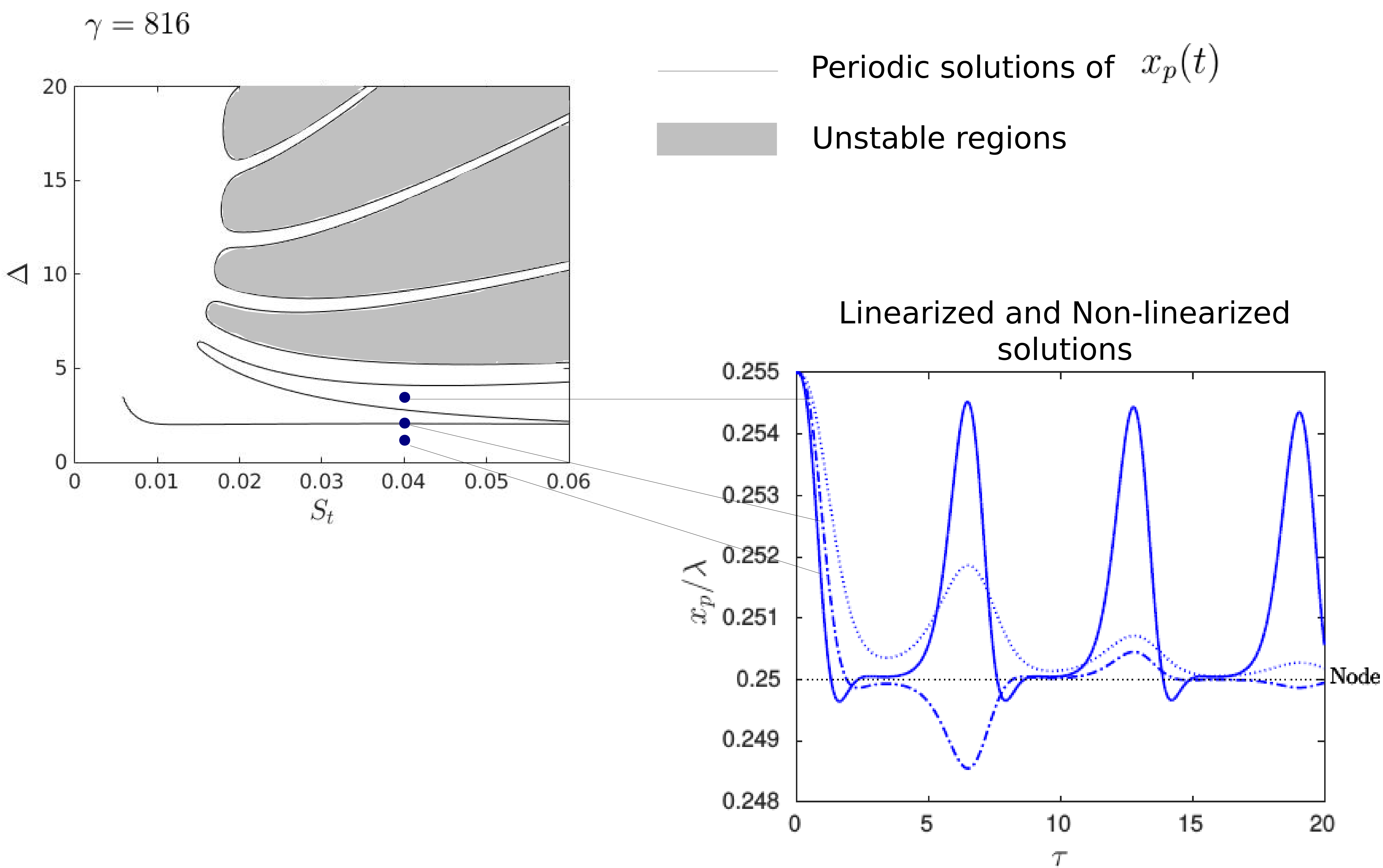
$$det(H_N) =$$

$$\begin{vmatrix}
\left(\frac{1}{4\pi^2 S_t}\right)^2 + \frac{N^2}{4} & 0 & \frac{i\Delta}{8\pi S_t} - \frac{\Delta}{2\gamma} & 0 & \frac{\Delta^2}{4\gamma} & 0 & 0 \\
0 & \left(\frac{1}{4\pi^2 S_t}\right)^2 + \frac{(N-1)^2}{4} & 0 & \frac{i\Delta}{8\pi S_t} - \frac{\Delta}{2\gamma} & 0 & \frac{\Delta^2}{4\gamma} & 0 \\
\dots & 0 & \dots & 0 & \dots & 0 & \dots \\
0 & \dots & 0 & \left(\frac{1}{4\pi^2 S_t}\right)^2 & 0 & \dots & 0 \\
\dots & 0 & \dots & 0 & \dots & .0 & \dots \\
0 & \frac{\Delta^2}{4\gamma} & 0 & -\frac{i\Delta}{8\pi S_t} - \frac{\Delta}{2\gamma} & 0 & \left(\frac{1}{4\pi^2 S_t}\right)^2 + \frac{(N-1)^2}{4} & 0 \\
0 & 0 & \frac{\Delta^2}{4\gamma} & 0 & -\frac{i\Delta}{8\pi S_t} - \frac{\Delta}{2\gamma} & 0 & \left(\frac{1}{4\pi^2 S_t}\right)^2 + \frac{N^2}{4}
\end{vmatrix}$$

Principal stability map

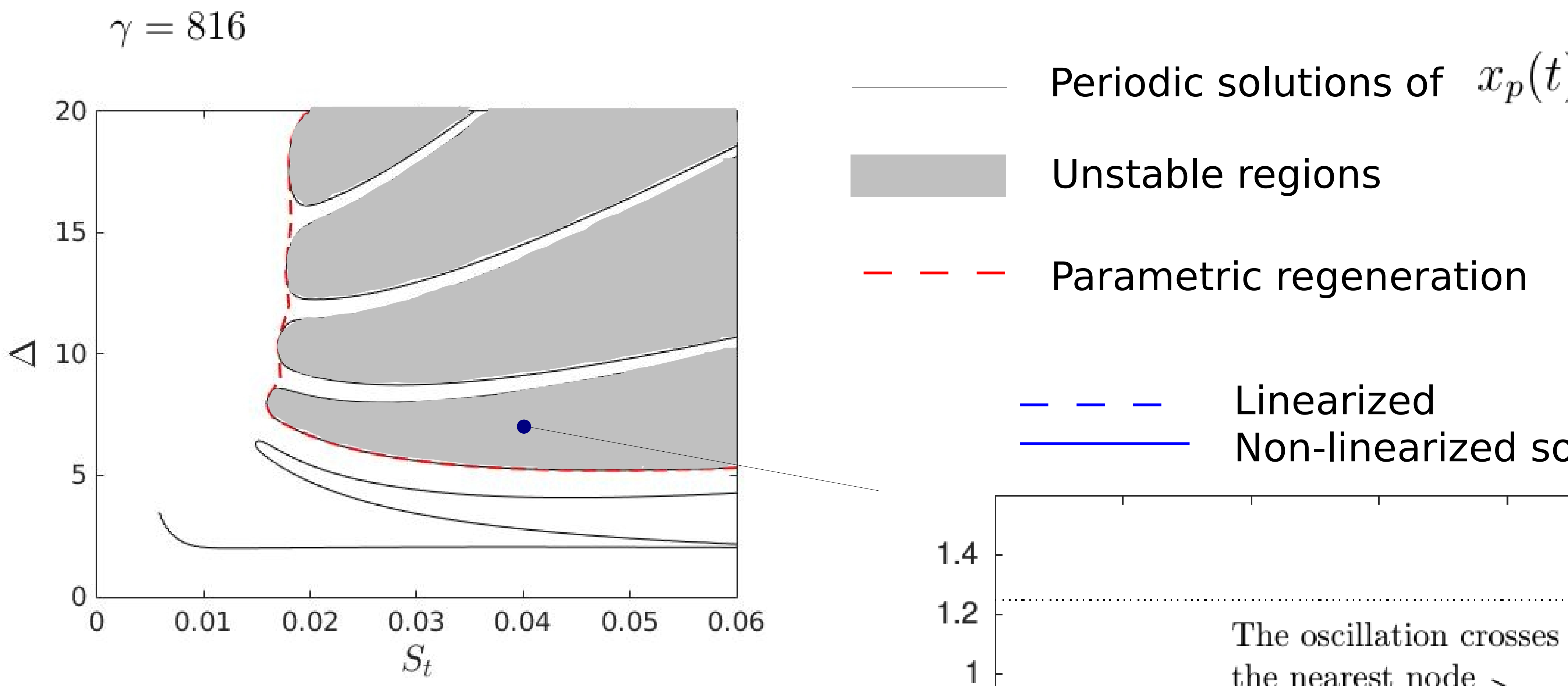


Validation



$\gamma = 816$
20
15
10
5
0
Δ
0
0.01
0.02
0.03
0.04
0.05
0.06
S_t
Periodic solutions of $x_p(t)$
Unstable regions
Linearized and Non-linearized solutions
0.255
0.254
0.253
0.252
0.251
0.25
0.249
0.248
x_p/λ
Node
0
5
10
15
20
τ

Validation


$\gamma = 816$
20
15
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Δ
0
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0.03
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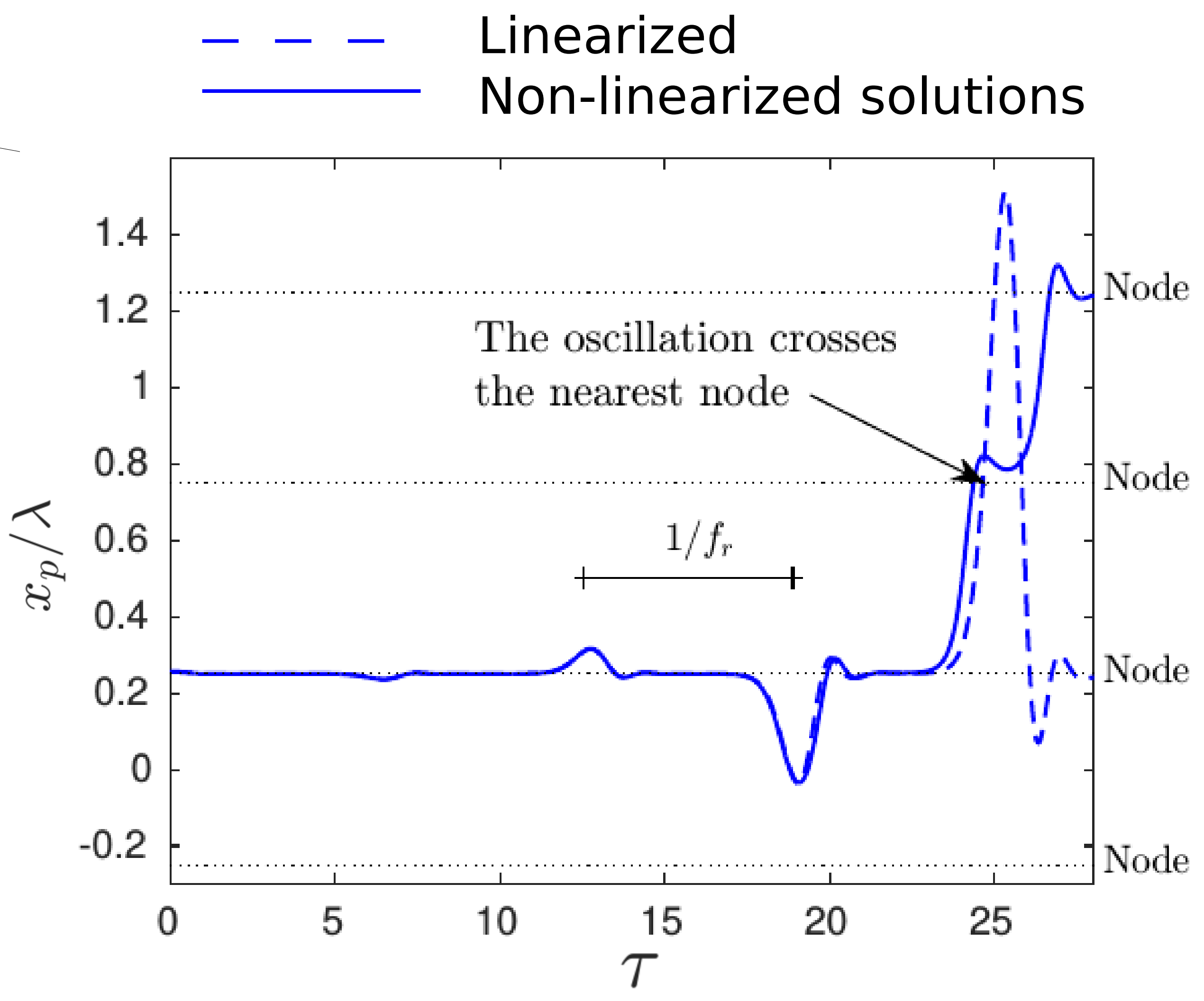
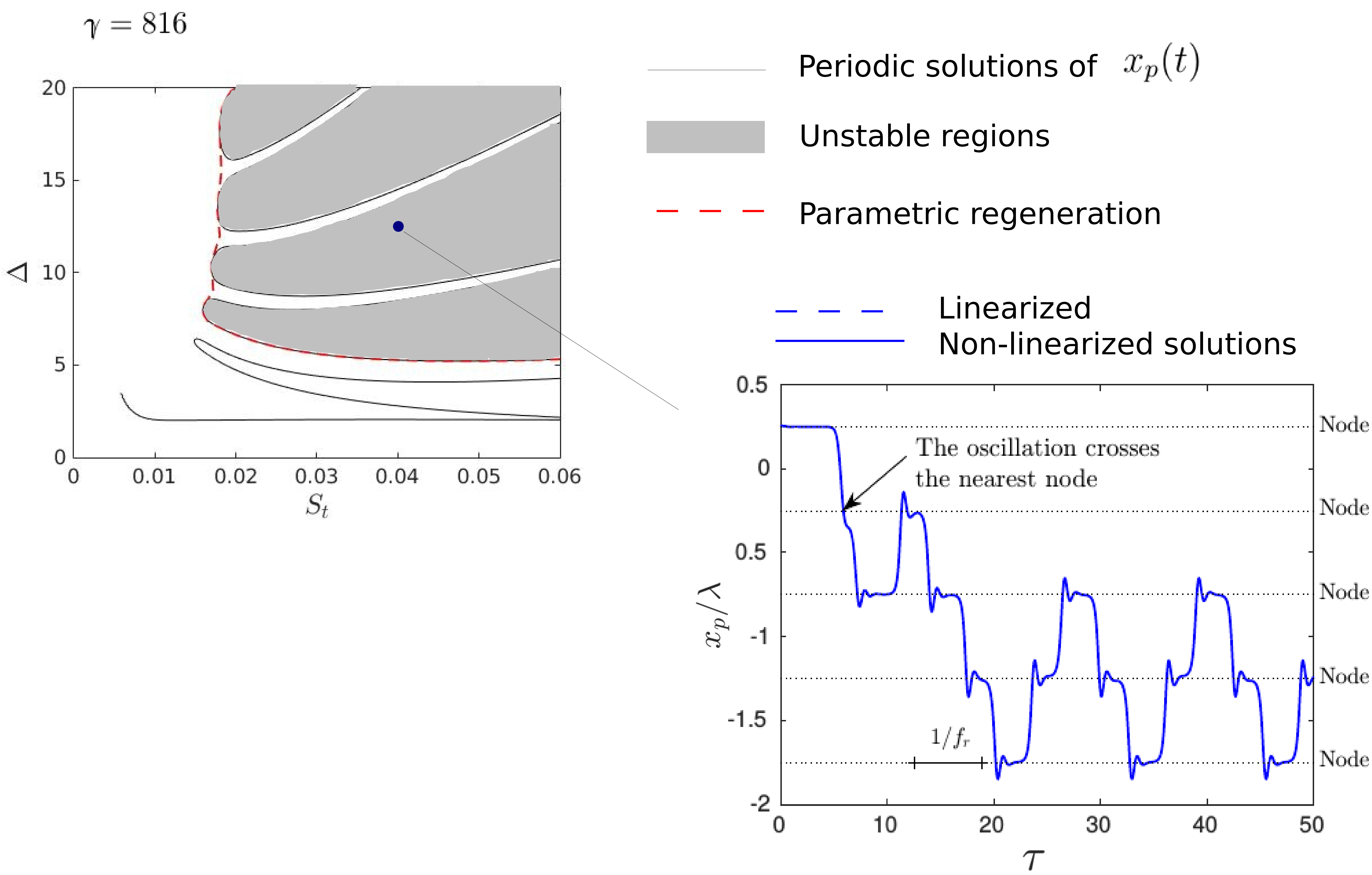
0.06
S_t
Periodic solutions of $x_p(t)$
Unstable regions
Parametric regeneration


Linearized
Non-linearized solutions
The oscillation crosses
the nearest node
$1/f_r$
Node
Node
Node
Node
x_p/λ
1.4
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
5
10
15
20
25
τ

Validation



$\gamma = 816$
Periodic solutions of $x_p(t)$
Unstable regions
Parametric regeneration
Linearized
Non-linearized solutions
20
15
10
5
0
Δ
0
0.01
0.02
0.03
0.04
0.05
0.06
S_t
0.5
0
0.5
-1
-1.5
-2
x_p/λ
0
10
20
30
40
50
τ
The oscillation crosses the nearest node
$1/f_r$
Node
Node
Node
Node
Node

Conclusions

- A parametric resonant particle response is demonstrated and the stability maps are constructed.
- The sub-harmonic response is dominant.
- Agglomeration can occur be still induced at different nodes even in a resonant response.

Thanks for your attention!

Time for questions